A secondary resonance in Mercury's rotation

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ORIGINAL ARTICLE

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Abstract The resonant rotation of Mercury can be modelised by a kernel model on which we can add perturbations. Our kernel model is a two-degree of freedom one written in Hamiltonian formalism. For this kernel, we consider that Mercury is solid and rotates on a Keplerian orbit. By introducing the perturbations due to the other planets of the Solar System, it appears that, in a particular case, our slow degree of freedom may enter into a 1:1 resonance with the Great Inequality of Jupiter and Saturn. Actually, as the moments of inertia of Mercury are still poorly known, this phenomenon cannot be excluded.

Keywords Mercury · Planetary rotation · Hamiltonian formalism · Resonance · Great Inequality · Kernel models

1 Introduction

The missions MESSENGER and BepiColombo being real catalysers for the research on Mercury, more and more precise studies and theories emerge until a few years. Let us cite a few of the most recent ones: Margot (2009), who presents revised values of the north pole orientation, Dufey et al. (2008), who model the planetary perturbations on Mercurys libration in longitude and D'Hoedt and Lemaître (2008), who show that Mercury stays in the Cassini forced state in an adiabatic way under the action of planetary long periodic terms.

However, shorter planetary periodic terms can, in some cases, induce a secondary resonance, it is for instance the case of Titan Noyelles (2008). To model this secondary resonance due to the influence of other planets of the Solar System in Mercury's rotation, we have to add the planetary perturbations to our kernel model. Our kernel model (D'Hoedt and Lemaître 2004) is a two-degree of freedom one written in Hamiltonian formalism. Moreover we make

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the following hypothesis: Mercury is a solid body, its orbit is Keplerian orbit, it rotates around its axis of greatest inertia and no dissipative forces acts on its movement. If we compare the free precession period of our model with the one of the Great Inequality of Jupiter and Saturn, they are clearly of the same order of magnitude (respectively 1,065.08 and 883.28 years). However, the 1,065.08 period computed from our model was based on the hypotheses that Mercury is a rigid body and if we introduce a liquid core as, e.g., Dufey et al. (2009); Peale (1976); Peale et al. (2008, 2009), we may obtain the exact equality between the two periods for a peculiar value of C_m/C where C is the polar moment of inertia of Mercury and C_m the one of its mantle. This assumption, even though theoretical, must not be excluded because this latest ratio is still poorly known. The missions Messenger and BepiColombo should give us a better knowledge of it (Milani et al. 2001).

In Sect. 2, we give a summary of our kernel rigid model and of the main results (equilibrium values, proper periods) obtained from it.

In Sect. 3, we consider that Mercury has a liquid core with the value of C_m/C needed to have a 1:1 resonance with the Great Inequality of Jupiter and Saturn that we introduce in the orbital elements thanks to Simon's series at first order.

In Sect. 4, we define the secondary resonant angle from the action-angle coordinates and we compute its proper period.

In Sect. 5, we analyse the influence of the secondary resonance on the main degrees of freedom.

2 Summary of the kernel model

In this model, we consider Mercury as a rigid body, we do not take into account the planetary perturbations and we assume that the Spin axis and the axis of greatest inertia are aligned. In this way, our two-degree of freedom Hamiltonian averaged on the mean anomaly can be written as follows (see D'Hoedt and Lemaître (2004) for details):

$$
\begin{aligned}
\langle\mathcal{H}\rangle = & -\frac{m^3\mu^2}{2\left(\Lambda_o-\frac{3\Lambda_1}{2}\right)^2}+\frac{\Lambda_1^2}{2C}-\frac{\mathcal{G}Mm^7\mu^3R_e^2}{(\Lambda_0-\frac{3}{2}\Lambda_1)^6}\\
& \times\left[\frac{1}{2}C_2^0\left(1+\frac{3e^2}{2}\right)\left(-\frac{1}{4}\left(-1+3\cos i_o^2\right)\left(-1+3\cos^2 K\right)\right.\right.\\
& -3\cos i_o\cos K\cos(\sigma_3)\sin i_o\sin K\\
& \left.-\frac{3}{4}\left(1-\cos^2 i_o\right)\left(1-\cos^2 K\right)\cos(2\sigma_3)\right)\\
& +3C_2^2\left(\frac{7e}{2}-\frac{123e^3}{16}\right)\left(\frac{1}{16}(1-\cos i_o)^2(1-\cos K)^2\cos(2\sigma_1+4\sigma_3)\right.\\
& +\frac{1}{4}(1-\cos i_o)(1-\cos K)\sin i_o\sin K\cos(2\sigma_1+3\sigma_3)\\
& +\frac{3}{8}\sin^2 i_o\sin^2 K\cos(2\sigma_1+2\sigma_3)\\
& +\frac{1}{4}(1+\cos i_o)(1+\cos K)\sin i_o\sin K\cos(2\sigma_1+\sigma_3)\\
& \left.\left.+\frac{1}{16}(1+\cos i_o)^2(1+\cos K)^2\cos(2\sigma_1)\right)\right]
\end{aligned}
\tag{1}
$$

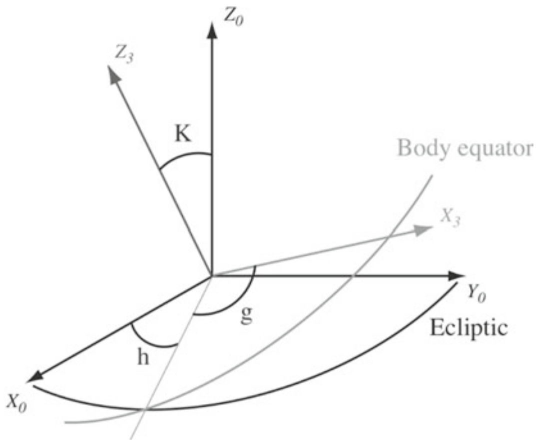


Fig. 1 The ecliptic frame ($\mathbf{X_0}, \mathbf{Y_0}, \mathbf{Z_0}$) with $\mathbf{X_0}$ and $\mathbf{Y_0}$ fixed in the ecliptic plane at a determined epoch and $\mathbf{Z_0}$ normal to the ecliptic plane and the body frame ($\mathbf{X_3}$,$\mathbf{Y_3}$,$\mathbf{Z_3}$) with $\mathbf{X_3}$ in the direction of the axis of biggest inertia and $\mathbf{Z_3}$ in the direction of the axis of smallest inertia of Mercury. h, g and K are the Euler's angles linking both frames

where e the eccentricity, i_o the orbital inclination, m the mass of Mercury, M the mass of the Sun, R_e the equatorial radius of Mercury, $\mathcal{G}$ the gravitational constant, $\mu = \mathcal{G}(m+M)$, Λ_o the conjugated momentum of the mean anomaly l_o and C being the greatest moment of inertia.

In this Hamiltonian, the first term comes from the two-body problem, the second one is the kinetic energy of rotation and the third big one is the potential of the gravity field. The variables and their conjugated momenta are

$$\begin{array}{ll} \sigma_1 = \frac{2(g+h)-3l_o}{2} - h_o - go & \Lambda_1 \\ \sigma_3 = -h + h_o & \Lambda_3 = \Lambda_1\,(1-\cos K) \end{array}$$

where g, h and K are Euler's angles linking the body frame to the ecliptic frame (see Fig. 1), h_o is the longitude of the ascending node, g_o is the pericenter argument and Λ_1 is the angular momentum norm.

σ_1 is thus the 3:2 spin-orbit resonant angle and K is called the ecliptic obliquity. The equilibrium values of this model found for the present state of Mercury were the following:

$$\begin{aligned} \sigma_1 &= 0, \\ \sigma_3 &= 0, \\ \Lambda_1 &= 13.303\frac{m\,R_e^2}{\text{year}}, \\ K &= 7^{\text{o}}, \end{aligned} \tag{2}$$

and the proper periods of angles were:

$$\begin{aligned} T_1 &= 15.8573\,\text{years}, \\ T_3 &= 1{,}065.08\,\text{years}. \end{aligned} \tag{3}$$

3 The Great Inequality of Jupiter and Saturn

By introducing the perturbations due to the other planets of the Solar System on our kernel model, it appears that, in a particular case, our slow degree of freedom may enter in a 1:1 resonance with the Great Inequality of Jupiter and Saturn. This was already underlined by

Dufey et al. (2009) who apply a perturbation theory based on the Lie triangle to re-introduce short periodic terms due to planetary perturbations into our averaged kernel Hamiltonian and so compute the evolution of the rotational variables (Dufey et al. 2008).

So, the period T_{JS} of the Great Inequality of Jupiter and Saturn, obtained thanks to Simon's series,[1] is

$$T_{JS} = 883.28 \text{ years.} \tag{4}$$

The proximity of the two periods T_3 and T_{JS} led us to look for a perfect equality for a more realistic model of Mercury with a liquid core. Effectively, if we take a value of $C_m/C = 0.82931$, this phenomenon may occur. However, according to (Margot et al. 2007), this value is not very probable but the parameters of Mercury are so poorly known that we must not exclude it (Figs. 5 and 6 in Dufey et al. 2009).

So, in this particular case, there is a 1:1 resonance between σ_3 and $\sigma_{25} = 2l_J - 5l_S$ where l_J is the mean anomaly of Jupiter and l_S the one of Saturn.

Let us thus introduce this angle σ_{25} in the orbital elements thanks to Simon's series at first order:

$$\begin{aligned}
e &= 0.206 + 3.03045\,10^{-8}\,\cos\sigma_{25} + 2.05414\,10^{-8}\,\sin\sigma_{25} &(5)\\
\cos i_o &= \cos 7^o + 9.37904\,10^{-10}\,\cos\sigma_{25} + 1.33666\,10^{-9}\,\sin\sigma_{25} &(6)\\
\sin i_o &= \sin 7^o - 7.63861\,10^{-9}\,\cos\sigma_{25} - 1.08862\,10^{-8}\,\sin\sigma_{25} &(7)\\
g_o &= 29.12478^o + 8.74537\,10^{-8}\,\cos\sigma_{25} - 4.94628\,10^{-7}\,\sin\sigma_{25} &(8)\\
l_o &= l_{o_0} + 5.71111\,10^{-8}\,\cos\sigma_{25} + 5.16281\,10^{-7}\,\sin\sigma_{25} &(9)\\
h_o &= 48.33^o + 5.62001\,10^{-8}\,\cos\sigma_{25} + 2.07223\,10^{-7}\,\sin\sigma_{25} &(10)
\end{aligned}$$

where l_{o_0} the mean anomaly without perturbations.

Our Hamiltonian can now be written:

$$\langle\mathcal{H}_{25}\rangle = -\frac{m^3\mu^2}{2\left(\Lambda_o - \frac{3\Lambda_1}{2}\right)^2} + \frac{\Lambda_1^2}{2C_m} + V_G(\sigma_1, \sigma_3, \sigma_{25}, \Lambda_1, \Lambda_3) + \nu\,\Lambda_{25} \tag{11}$$

with ν the frequency and Λ_{25} the conjugated momentum of σ_{25}. Let us note that the last term has to be added in order to keep an autonomous Hamiltonian. The complete expression of G is given in the Appendix.

4 Secondary resonant angle

In order to define our secondary resonant angle, we have to express our Hamiltonian in action-angle coordinates.

To do this, we have to perform a succession of transformations (D'Hoedt and Lemaître 2006):

- a canonical transformation into cartesian coordinates,
- a translation to the equilibrium,
- a Mac Laurin's expansion up to the order 2 for our first degree of freedom and up to the fourth order for our third degree of freedom,

[1] These series were given by J.-L. Simon in a private communication and are obtained from the VSOP theory (Fienga and Simon 2005).

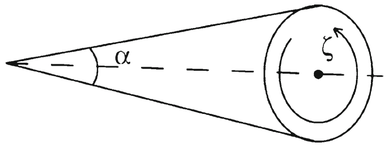


Fig. 2 During ζ performs a complete run, α oscillates between its minimum and maximum values

– an untangling transformation (Henrard and Lemaître 2005) to eliminate the mixed terms,
– a scale transformation to associate to each variable and its conjugate momentum the same coefficient,
– a transformation into action-angle coordinates $(J_1, J_3, \psi_1, \psi_3)$ where ψ_1 and ψ_3 are the angles of free libration around the equilibrium.

Let us note that we had to develop up to the fourth order to have terms in J_3^2.

We can now define our 1:1 secondary resonant angle

$$\alpha = \psi_3 - \sigma_{25}. \tag{12}$$

Our canonical set of action-angle variables is thus now

$$\begin{array}{ll} \psi_1 & J_1 \\ \alpha & J_3 \\ \sigma_{25} & \Lambda'_{25} = J_3 + \Lambda_{25}. \end{array}$$

The first degree of freedom (J_1, ψ_1) being of no interest for the calculation of the period of our secondary resonant angle, we consider it as constant and drop it.

After that simplification, we average our Hamiltonian about σ_{25} (Λ_{25} becomes thus a constant) and obtain this expression:

$$\begin{aligned} \langle \mathcal{H}_{25} \rangle = & -0.000184\ J_3^2 + 5.50471\ 10^{-10} \sqrt{J_3} \cos \alpha \\ & -2.97662\ 10^{-10} \sqrt{J_3} \sin \alpha. \end{aligned} \tag{13}$$

We again compute the equilibria of this Hamiltonian, perform a variables change to centre the Hamiltonian at the equilibrium and expand it in Mac Laurin's series up to the second order.

After a scaling transformation and an action-angle transformation, the final form of the Hamiltonian is

$$\begin{aligned} \langle \mathcal{H}_{25} \rangle = & -5.727\ 10^{-8}\ Z + 2.307\ 10^{-5}\ Z^2 - 1.15\ 10^{-6}\ Z^{3/2} \cos \zeta \\ & +1.15\ 10^{-}6\ Z^{3/2} \cos 3\zeta - 2.307\ 10^{-5}\ Z^2 \cos 4\zeta. \end{aligned} \tag{14}$$

The frequency of ζ being the coefficient of Z in (14), we can deduce the period of ζ and thus the proper period of the free librations of α:

$$T_\alpha = 1.0972\ 10^8 \text{ years}. \tag{15}$$

Actually, the period of ζ is also the one of the movement of α because during a complete run of ζ around its equilibrium, α performs one going and coming between its minimum and maximum values of libration (see Fig. 2).

5 Influence of the secondary resonance on the other degrees of freedom

If we compute and numerically integrate the equations of motion obtained from (11), we can see the influence of the secondary resonance on our principal degrees of freedom.

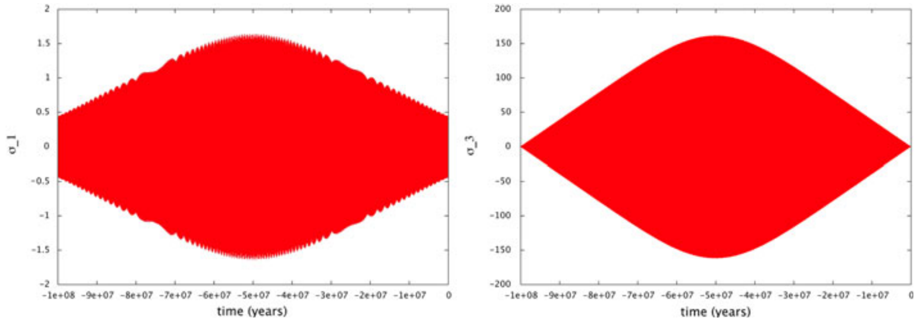
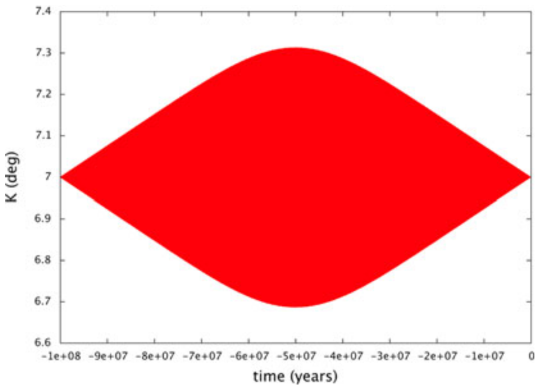


Fig. 3 Influence of the Great Inequality on σ_1 and σ_3

Fig. 4 Influence of the Great Inequality on the ecliptic obliquity K (in degrees)



As expected, since the resonance is between σ_3 and the Great Inequality, its influence on σ_3 and K is much bigger than on σ_1: on σ_1, the amplitude is of 1.64 arc sec (to be compared with a maximum amplitude of libration of 40 arc sec Dufey et al. 2009), while it is of 161.49 arc sec on σ_3 (Fig. 3) and of $0.3132° = 18.79$ arc min on the ecliptic obliquity K (Fig. 4). This amplitude is especially large because if we examine the others effects that can introduce a forced libration on the ecliptic obliquity, we can see the most important of them is due to the precession of the ascending node of Mercury for which the amplitude of the variations are only of 0.414 arsec on a period of 63,315 years (Dufey et al. 2009). However, in both cases, the periods being very long, all what could be observed is constant offsets.

6 Conclusion

Due to the proximity of the period of one of our degree of freedom and of the Great Inequality of Jupiter and Saturn, we built a theoretical but nevertheless possible (with a weak probability) model taking into account this 1:1 secondary resonance. Even if the influence of this resonance on our main degrees of freedom is non negligible concerning the amplitudes, the proper period is so long that it could only be observed as a constant offset.

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Appendix: Complete expression of V_G

$$
\begin{aligned}
V_G = & -\frac{\mathcal{G}\, m^7\, M\, R_e^2\, \mu^3}{(\Lambda_0 - \frac{3\Lambda_1}{2})^6}\Bigg[-\frac{3}{2}C_2^0\ (0.536 + 0.52\cos^2 K + 8.939\ 10^{-9}\cos\sigma_{25} \\
& + 1.064\ 10^{-8}\cos^2 K\cos\sigma_{25} + 0.004\cos 2\sigma_3 - 0.004\cos^2 K\cos\sigma_3 \\
& - 4.256\ 10^{-10}\cos\sigma_{25}\cos 2\sigma_3 + 4.256\ 10^{-10}\cos^2 K\cos\sigma_{25}\cos 2\sigma_3 \\
& + 0.129\cos K\cos\sigma_3\sin K - 5.677\ 10^{-9}\cos K\cos\sigma_{25}\cos\sigma_3\sin K \\
& + 5.689\ 10^{-9}\sin\sigma_{25} + 8.323\ 10^{-9}\cos^2 K\sin\sigma_{25} \\
& - 6.584\ 10^{-10}\cos 2\sigma_3\sin\sigma_{25} + 6.584\ 10^{-10}\cos^2 K\cos 2\sigma_3\sin\sigma_{25} \\
& - 9.784\ 10^{-9}\cos K\cos\sigma_3\sin K\sin\sigma_{25} \\
& + 7.231\ 10^{-9}\cos K\cos\sigma_{25}\sin K\sin\sigma_3 \\
& - 2.666\ 10^{-8}\cos K\sin K\sin\sigma_{25}\sin\sigma_3 + 4.439\ 10^{-10}\cos\sigma_{25}\sin 2\sigma_3 \\
& - 4.439\ 10^{-10}\cos^2 K\cos\sigma_{25}\sin 2\sigma_3 - 1.637\ 10^{-9}\sin\sigma_{25}\sin 2\sigma_3 \\
& + 1.637\ 10^{-9}\cos^2 K\sin\sigma_{25}\sin 2\sigma_3) \\
& + 3\ C_2^2(0.162\cos\sigma_1 + 0.3245\cos K\cos 2\sigma_1 \\
& + 0.162\cos^2 K\cos 2\sigma_1 + 1.911\ 10^{-8}\cos 2\sigma_1\cos\sigma_{25} \\
& + 3.823\ 10^{-8}\cos K\cos 2\sigma_1\cos\sigma_{25} + 1.911\ 10^{-8}\cos^2 K\cos 2\sigma_1\cos\sigma_{25} \\
& + 0.004\cos(2\sigma_1 + \sigma_3) - 0.004\cos^2 K\cos(2\sigma_1 + 2\sigma_3) \\
& + 2.27\ 10^{-6}\cos(2\sigma_1 + 4\sigma_3) - 4.541\ 10^{-6}\cos K\cos(2\sigma_1 + 4\sigma_3) \\
& + 2.27\ 10^{-6}\cos^2 K\cos(2\sigma_1 + 4\sigma_3) + 0.04\cos(2\sigma_1 + \sigma_3)\sin K \\
& + 0.04\cos K\cos(2\sigma_1 + \sigma_3)\sin K + 2.169\ 10^{-9}\cos\sigma_{25}\cos(2\sigma_1 + \sigma_3)\sin K \\
& + 2.169\ 10^{-9}\cos K\cos\sigma_{25}\cos(2\sigma_1 + \sigma_3)\sin K \\
& + 0.0002\cos(2\sigma_1 + 3\sigma_3)\sin K - 0.0002\cos K\cos(2\sigma_1 + 3\sigma_3)\sin K \\
& + 3.794\ 10^{-8}\cos\sigma_{25}\sin 2\sigma_1 + 7.587\ 10^{-8}\cos K\cos\sigma_{25}\sin 2\sigma_1 \\
& + 3.794\ 10^{-8}\cos^2 K\cos\sigma_{25}\sin 2\sigma_1 + 1.307\ 10^{-8}\cos 2\sigma_1\sin\sigma_{25} \\
& + 2.614\ 10^{-8}\cos K\cos[2\sigma_1\sin\sigma_{25} + 1.307\ 10^{-8}\cos^2 K\cos 2\sigma_1\sin\sigma_{25} \\
& - 3.621\ 10^{-10}\cos(2\sigma_1 + 2\sigma_3)\sin\sigma_{25} \\
& + 3.621\ 10^{-10}\cos^2 K\cos(2\sigma_1 + 2\sigma_3)\sin\sigma_{25} \\
& - 3.747\ 10^{-10}\cos(2\sigma_1 + \sigma_3)\sin K\sin\sigma_{25} \\
& - 3.747\ 10^{-10}\cos K\cos(2\sigma_1 + \sigma_3)\sin K\sin\sigma_{25} \\
& + 1.58\ 10^{-7}\sin 2\sigma_1\sin\sigma_{25} + 3.16\ 10^{-7}\cos K\sin 2\sigma_1\sin\sigma_{25} \\
& + 1.58\ 10^{-7}\cos^2 K\sin 2\sigma_1\sin\sigma_{25} + 1.151\ 10^{-8}\cos\sigma_{25}\sin K\sin(2\sigma_1 + \sigma_3) \\
& + 1.151\ 10^{-8}\cos K\cos\sigma_{25}\sin K\sin(2\sigma_1 + \sigma_3) \\
& + 3.044\ 10^{-8}\sin K\sin\sigma_{25}\sin(2\sigma_1 + \sigma_3) \\
& + 3.044\ 10^{-8}\cos K\sin K\sin\sigma_{25}\sin(2\sigma_1 + \sigma_3) \\
& + 1.261\ 10^{-9}\cos\sigma_{25}\sin(2\sigma_1 + 2\sigma_3) \\
& - 1.261\ 10^{-9}\cos^2 K\cos\sigma_{25}\sin(2\sigma_1 + 2\sigma_3) + 2.0377\ 10^{-9}\sin\sigma_{25}\sin(2\sigma_1 + 2\sigma_3) \\
& - 2.038\ 10^{-9}\cos^2 K\sin\sigma_{25}\sin(2\sigma_1 + 2\sigma_3))\Bigg]
\end{aligned}
\tag{16}
$$

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